

Solutions to Practice Problems with Assignment #5

II. Semantic Proofs

A. Logical Truth

1. Prove “ $(F(a,b) \rightarrow \neg(G(d,e,c) \wedge \neg F(a,b)))$ ” is L-true.

Proof: Suppose not, i.e., suppose that some interpretation, I, makes

“ $(F(a,b) \rightarrow \neg(G(d,e,c) \wedge \neg F(a,b)))$ ” false. Also, for some $\alpha \in D$, “a” designates α on I, and for some $\beta \in D$, “b” designates β on I. Then

1. I makes “ $F(a,b)$ ” true AND

2. I makes “ $\neg(G(d,e,c) \wedge \neg F(a,b))$ ” false. [TC $\rightarrow$]

3. From 1., $\langle \alpha, \beta \rangle \in “F”$ on I. [TC Atomic]

4. From 2., I makes “ $(G(d,e,c) \wedge \neg F(a,b))$ ” true. [TC $\neg$]

5. From 4., I makes “ $G(d,e,c)$ ” true AND I makes “ $\neg F(a,b)$ ” true. [TC $\wedge$]

6. From 5., I makes “ $F(a,b)$ ” false. [TC $\neg$].

7. So, I makes $\langle \alpha, \beta \rangle \notin “F”$. [TC Atomic]

But this means that $\langle \alpha, \beta \rangle \in “F”$ on I (from 3) AND $\langle \alpha, \beta \rangle \notin “F”$ on I (from 7). That’s impossible. So, no I makes “ $(F(a,b) \rightarrow \neg(G(d,e,c) \wedge \neg F(a,b)))$ ” false. Hence, it is a logical truth.

2. Homework!

3. Prove “ $((((A(a) \wedge B(b)) \vee (A(a) \wedge \neg B(b))) \vee (\neg A(a) \wedge B(b))) \vee (\neg A(a) \wedge \neg B(b)))$ ” is L-true.

Proof: Suppose not, i.e., suppose that some I makes

“ $((((A(a) \wedge B(b)) \vee (A(a) \wedge \neg B(b))) \vee (\neg A(a) \wedge B(b))) \vee (\neg A(a) \wedge \neg B(b)))$ ” false.

Also, for some $\alpha \in D$, “a” designates α on I and for some $\beta \in D$, “b” designates β on I.

1. So, I makes “ $((((A(a) \wedge B(b)) \vee (A(a) \wedge \neg B(b))) \vee (\neg A(a) \wedge B(b))) \vee (\neg A(a) \wedge \neg B(b)))$ ” false AND

2. I makes “ $(\neg A(a) \wedge \neg B(b))$ ” false. [TC $\vee$]

3. From 2., I makes “ $\neg A(a)$ ” false OR “ $\neg B(b)$ ” false. [TC $\wedge$]

4. From 3. I makes “ $A(a)$ ” true OR makes “ $B(b)$ ” true. [TC $\neg$]

5. From 4., $\alpha \in “A”$ on I OR $\beta \in “B”$ on I. [TC Atomic]

6. From 1., I makes “ $((A(a) \wedge B(b)) \vee (A(a) \wedge \neg B(b)))$ ” false AND

7. I makes “ $(\neg A(a) \wedge B(b))$ ” false. [TC $\vee$]

8. From 7., I makes “ $\neg A(a)$ ” false OR “ $B(b)$ ” false. [TC $\wedge$]

9. From 8., I makes “ $A(a)$ ” true. [TC $\neg$]

10. From 8. and 9., $\alpha \in “A”$ on I OR $\beta \notin “B”$ on I. [TC Atomic]

11. From 6., I makes “ $((A(a) \wedge B(b)))$ ” false AND

12. I makes “ $(A(a) \wedge \neg B(b))$ ” false. [TC $\vee$]

13. From 12., I makes “ $A(a)$ ” false OR “ $\neg B(b)$ ” false. [TC $\wedge$]

14. From 13., I makes “ $B(b)$ ” true. [TC $\neg$]

15. So, from 13. and 14., $\alpha \notin “A”$ on I OR $\beta \in “B”$ on I. [TC Atomic]

16. From 11., I makes “ $A(a)$ ” false OR “ $B(b)$ ” false. [TC $\wedge$]

17. From 16., $\alpha \notin “A”$ on I OR $\beta \notin “B”$ on I. [TC Atomic]

18. Grouping the results, we get (from 5.) $\alpha \in “A”$ on I OR $\beta \in “B”$ on I, AND (from 10.)

$\alpha \in “A”$ on I OR $\beta \notin “B”$ on I, AND (from 15.) $\alpha \notin “A”$ on I OR $\beta \in “B”$ on I, AND (from 17.)

$\alpha \notin “A”$ on I OR $\beta \notin “B”$ on I. So, there are 4 cases of objects and predicate to consider.

CASE 1: Suppose $\alpha \in \text{"A"}$ on I. From this supposition and 15. We get $\beta \in \text{"B"}$ on I.
 But, from $\beta \in \text{"B"}$ on I and 17., $\alpha \notin \text{"A"}$ on I. That means that if we suppose $\alpha \in \text{"A"}$ on I, we get $\alpha \notin \text{"A"}$ on I. That's impossible.

CASE 2: Suppose $\alpha \notin \text{"A"}$ on I. From this supposition and 10., we get that $\beta \notin \text{"B"}$ on I.
 But from $\beta \notin \text{"B"}$ on I and 5., we get $\alpha \in \text{"A"}$ on I. That's impossible.

CASE 3: Suppose $\beta \in \text{"B"}$ on I. From this supposition and 10., we get $\alpha \in \text{"A"}$ on I.
 But from $\alpha \in \text{"A"}$ on I and 17., we get $\beta \notin \text{"B"}$ on I. That's impossible.

CASE 4: Suppose $\beta \notin \text{"B"}$ on I. From this supposition and 5., we get $\alpha \in \text{"A"}$ on I.
 But from $\alpha \in \text{"A"}$ on I and 15., we get $\beta \in \text{"B"}$ on I. That's impossible.

But these are all of the possibilities for this supposed interpretation. Hence, the interpretation is impossible. So, no interpretation makes the original sentence false. Thus, it is logically true.

B. Logical Falsehood

1. Homework!

2. Prove " $((F(a) \wedge \neg F(b)) \wedge (a = b))$ " is L-false.

Proofs. Suppose not, i.e., suppose some I makes " $((F(a) \wedge \neg F(b)) \wedge (a = b))$ " true. Also, for some $\alpha \in D$, "a" designates α on I and for some $\beta \in D$, "b" designates β on I. So, I makes

1. " $((F(a) \wedge \neg F(b))$ " true AND
2. " $(a = b)$ " true.
3. From 1., I makes " $F(a)$ " true AND
4. I makes " $\neg F(b)$ " true. [TC $\wedge$]
5. From 4., I makes " $F(b)$ " false. [TC $\neg$]
6. From 3., $\alpha \in \text{"F"}$ on I. [TC Atomic]
7. From 5., $\beta \notin \text{"F"}$ on I. [TC Atomic]
8. From 2. and 7., $\alpha \notin \text{"F"}$ on I. [Substitution] But, given 6., this is impossible. So, no I makes this sentence true. So it is logically false.

C. Entailment

1. $\{(F(a) \wedge F(b)), \neg F(b)\} \models G(a)$

Proof. Suppose not, i.e., suppose some I makes:

1. " $(F(a) \wedge F(b))$ " true
2. " $\neg F(b)$ " true
3. " $G(a)$ " false.

Also, for some $\alpha \in D$, "a" designates α on I, and for some $\beta \in D$, "b" designates β on I.

4. From 1., I makes " $F(a)$ " true AND makes " $F(b)$ " true. [TC $\wedge$]
5. From 2., I makes " $F(b)$ " false. [TC $\neg$] [Technically, this already contradicts 4. But...]
6. From 4., $\alpha \in \text{"F"}$ on I AND $\beta \in \text{"F"}$ on I. [TC Atomic]
7. From 5., $\beta \notin \text{"F"}$ on I. But given 6., that's impossible. So, no interpretation makes all the lhs sentence true. So no I makes all the lhs true and the rhs false. So, the entailment holds.

2. $\{(F(a,b) \vee G(c,d)), (F(a,b) \rightarrow A(a)), (G(c,d) \rightarrow B(b))\} \models (A(a) \vee B(b))$

Proof: Suppose not, i.e., suppose some I makes:

1. “ $(F(a,b) \vee G(c,d))$ ” true
2. “ $(F(a,b) \rightarrow A(a))$ ” true
3. “ $(G(c,d) \rightarrow B(b))$ ” true
4. “ $(A(a) \vee B(b))$ ” false.

Also, for some $\alpha \in D$, “a” designates α on I, and for some $\beta \in D$, “b” designates β on I, and for some $\gamma \in D$, “c” designates γ on I, and for some $\delta \in D$, “d” designates δ on I.

5. From 1., I makes “ $F(a,b)$ ” true OR makes “ $G(c,d)$ ” true. [TC $\vee$]
6. From 5., either $\langle \alpha, \beta \rangle \in “F”$ on I OR $\langle \gamma, \delta \rangle \in “G”$ on I. [TC Atomic]
7. From 2., either I makes “ $F(a,b)$ ” false OR I makes “ $A(a)$ ” true. [TC $\rightarrow$]
8. From 7., $\langle \alpha, \beta \rangle \notin “F”$ on I OR $\alpha \in “A”$ on I. [TC Atomic]
9. From 3., either I makes “ $G(c,d)$ ” false OR I makes “ $B(b)$ ” true. [TC $\rightarrow$]
10. From 9., either $\langle \gamma, \delta \rangle \notin “G”$ on I OR $\beta \in “B”$ on I. [TC Atomic]
11. From 4., I makes “ $A(a)$ ” false AND I makes “ $B(b)$ ” false. [TC $\vee$]
12. From 11., $\alpha \notin “A”$ on I AND $\beta \notin “B”$ on I. [TC Atomic]

From 12. and 8., we get $\langle \alpha, \beta \rangle \notin “F”$ on I. From 12. and 10. we get $\langle \gamma, \delta \rangle \notin “G”$ on I.

The first contradicts the first option from 6. The second contradicts the second option from 6. So neither option from 6. is possible. and 13. Thus no interpretation makes all the lhs sentences true and the rhs sentence false. Hence, the entailment holds.

D. Inconsistency. Show that the following sets are inconsistent.

1. $\{ (A(a) \vee A(b)), (A(a) \rightarrow G(c)), (G(c) \rightarrow \neg A(b)), (A(b) \rightarrow G(c)), (G(c) \rightarrow \neg A(a)) \}$

Proof: Suppose not, i.e., suppose some I makes:

1. “ $(A(a) \vee A(b))$ ” true
2. “ $(A(a) \rightarrow G(c))$ ” true
3. “ $(G(c) \rightarrow \neg A(b))$ ” true
4. “ $(A(b) \rightarrow G(c))$ ” true
5. “ $(G(c) \rightarrow \neg A(a))$ ” true

Also, for some $\alpha \in D$, “a” designates α on I, and for some $\beta \in D$, “b” designates β on I, and for some $\gamma \in D$, “c” designates γ on I.

6. From 1., I makes “ $A(a)$ ” true OR “ $A(b)$ ” true. [TC $\vee$]
7. From 6., either $\alpha \in “A”$ on I OR $\beta \in “A”$ on I. [TC Atomic]
8. From 2., I makes “ $A(a)$ ” false OR “ $G(c)$ ” true. [TC $\rightarrow$]
9. From 8., either $\alpha \notin “A”$ on I OR $\gamma \in “G”$ on I. [TC Atomic]
10. From 3., I makes “ $G(c)$ ” false OR “ $\neg A(b)$ ” true. [TC $\rightarrow$]
11. From 10., I makes “ $G(c)$ ” false OR “ $A(b)$ ” false. [TC $\neg$]
12. From 11., $\gamma \notin “G”$ on I OR $\beta \notin “A”$ on I. [TC Atomic]
13. From 4., I makes “ $A(b)$ ” false OR makes “ $G(c)$ ” true. [TC $\rightarrow$]
14. From 13., $\beta \notin “A”$ on I OR $\gamma \in “G”$ on I. [TC Atomic]
15. From 5., I makes “ $G(c)$ ” false OR makes “ $\neg A(a)$ ” true. [TC $\rightarrow$]
16. From 15., I makes “ $G(c)$ ” false OR makes “ $A(a)$ ” false. [TC $\neg$]
17. From 16., $\gamma \notin “G”$ on I OR $\alpha \notin “A”$ on I. [TC Atomic]

From 7., there are two cases to consider.

Case 1: Suppose $\alpha \in "A"$ on I. Then $\gamma \in "G"$ on I (by 9.). But then (by 17.) $\alpha \notin "A"$ on I. So this case is impossible.

Case 2: Suppose $\beta \in "A"$ on I. Then $\gamma \in "G"$ on I (by 14.). But then (by 12.) $\beta \notin "A"$ on I. So this case is impossible.

But Case 1 and Case 2 are the only ways to make the first sentence of the set true. Since both lead to contradiction no interpretation can make all of the sentences true. So, the set is inconsistent.

2. Homework!

3. $\{ (a = b), (b = c), (A(a) \wedge \neg A(c)) \}$

Proof. Suppose not, i.e., suppose that some I makes:

1. " $(a = b)$ " true

2. " $(b = c)$ " true

3. " $(A(a) \wedge \neg A(c))$ " true

Also, for some $\alpha \in D$, " a " designates α on I.

4. From 1., " b " designates α on I. [TC =]

5. From 4. and 2., we know that " c " designates α on I. [TC =]

6. From 3., I makes " $A(a)$ " true AND

7. I makes " $\neg A(c)$ " true. [TC $\wedge$]

8. From 7., I makes " $A(c)$ " false. [TC $\neg$]

9. From 6., $\alpha \in "A"$ on I. [TC Atomic]

10. From 8. and 5., $\alpha \notin "A"$ on I. [TC Atomic]

But 9. and 10. contradict, so this is impossible. So, no I makes all the sentences true. So the set of sentences is inconsistent.